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6 ANALYSIS OF A TWO-ECHELON INVENTORY
SYSTEM IN WHICH ALL LOCATIONS FOLLOW
CONTINUOUS REVIEW (S,s) POLICIES.

By

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ANALYSIS OF A TWO-ECHELON INVENTORY SYSTEM IN WHICH ALL LOCATIONS FOLLOW CONTINUOUS REVIEW (S,s) POLICIES

by

John A. Muckstadt

Page 13, line 8 should read:

$$= \begin{cases} 0; & \text{if } [(d+Q+r-k)/Q] + g \leq I_0/Q = q \quad \text{or } d \notin N_1^* \text{ when } i \geq k \\ & \text{and } d \notin N_2^* \text{ when } i < k \\ 1 \cdot a(\lambda R, d) \cdot a(\gamma R, g) \cdot \frac{1}{Q} \cdot \frac{1}{M}; & \text{otherwise,} \end{cases}$$

where $N_1^* = \{n: n = Q - (i-k) + \bar{n}Q, \bar{n} = 0, 1, 2, \dots\} \cup \{0\}$ and

$N_2^* = \{n: n = k - i + \bar{n}Q, \bar{n} = 0, 1, 2, \dots\}$.

Page 13, line 12: period after otherwise.

Page 13, line 12: $d \notin N_1$ and $d \notin N_2$ should be $d \notin N_1^*$ and $d \notin N_2^*$ respectively.

Page 14, line 14: Add $i = r+1, \dots, r+Q$ after $P(I^2 = i; \text{Case B})$.

Page 15, line 1 and line 3: $\sum_{n \in N_1}$ and $\sum_{n \in N_2}$ should be $\sum_{n \in N_1^*}$ and $\sum_{n \in N_2^*}$ respectively.

Page 20, line 8: $d_1 \in N_1$ and $d_1 \in N_2$ should be $d_1 \in N_1^*$ and $d_1 \in N_2^*$ respectively.

Page 20, line 10: N_1 and N_2 should be N_1^* and N_2^* respectively.


Page 21, line 1 and line 2: $\sum_{n \in N_1}$ and $\sum_{n \in N_2}$ should be $\sum_{n \in N_1^*}$ and $\sum_{n \in N_2^*}$ respectively.

Page 21, line 4: $H = \{d: d = 0, 1, \dots, [(n-k+Q+r)/Q] \cdot Q + Q(d-j+Q+r)/Q \cdot Q > i_0\}$.
should be $H = \{d: d = 0, 1, \dots; [(n-k+Q+r)/Q] \cdot Q + [(d-j+Q+r)/Q] \cdot Q > i_0\}$.


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SUMMARY

 ~~in~~ this paper ~~we~~ study^{ies} the probabilistic behavior of a two-echelon inventory system consisting of a depot and a set of bases. Primary demands occur at the bases for a single unit at a time. Whenever a base's inventory position reaches the reorder point (s) , the base orders sufficient inventory from the depot to raise the base's inventory position to S . Similarly, the depot places an order to its supplier when its inventory position reaches its reorder point; the order is for the number of units required to raise the depot's inventory position to a prespecified level. Thus all locations follow a continuous review (S,s) policy. All excess demand is assumed to be backordered.

~~Our~~^{The} main objective is to derive the probability distribution for the number of backordered units at a base at an arbitrary point in time given an item follows a known (S,s) policy at each base and the depot. The demand process at each base is assumed to be a stationary Poisson process. The analysis is carried out for two cases. In the first case, ~~it is~~^{it is} ~~we~~ assumed the system consists of a large number of bases; in the second case, ~~we~~^{it is} assumed there are two bases in the system. To simplify the discussion, ~~we~~^{it is} assumed in both cases that all bases follow the same (S,s) policy.



I. INTRODUCTION

In this paper we study the probabilistic behavior of an inventory system consisting of a depot (the first echelon), and a set of bases (the second echelon). Primary demands occur only at the bases. Each demand at a base is for a single unit of an item. When a base's inventory position--on-hand plus on-order minus backorders--reaches the reorder point (s) , the base orders enough inventory from the depot to raise the base's inventory position to a pre-established level (S) . Similarly, whenever the depot inventory position reaches its reorder point, the depot orders a sufficient quantity from its supplier to raise its inventory position to a given value. Thus each location in the system follows a continuous review (S,s) policy. Furthermore, demands occurring when a base is out of stock are assumed to be back-ordered.

Numerous real inventory systems operate in this manner. For example, the Air Force maintains bases throughout the world. Demand for spare parts occurs at these bases; the bases in turn are resupplied for most components by an Air Force or other Department of Defense depot. For each component there is only one depot that resupplies a base. Each location in this system follows a continuous review (S,s) type inventory policy.

Our main objective is to develop the probability distribution for the number of backordered units at a base at an arbitrary point in time for a specific item given a particular (S,s) policy for each base and the depot. The approach we take can also be used to determine the probability distribution for on-hand inventory at a base at an arbitrary

point in time. We assume the demand process at each base is a stationary Poisson process. The analysis is carried out for two cases.

In the first case, we assume the item is used at a large number of bases. For example, Xerox has many spare parts stocked by most of the approximately 8000 people who repair Xerox machines. Each person, corresponding to a base, is normally resupplied by one Branch warehouse, corresponding to a depot. Roughly 90 people are resupplied by a single Branch warehouse. To simplify the discussion, we will assume the demand process is the same at each base. Furthermore, the costs--carrying costs, shortage costs, and fixed and variable ordering costs--and depot-to-base transportation times are assumed to be the same for each base. If each base follows an (S,s) policy, they should all select the same values for S and s ; we assume that they do. (The analysis we present can be modified using methods similar to those used in Chapter 5 of reference 8 to study the case where demand distributions, costs, or transportation times are not the same at all bases).

Next, we examine a system consisting of two identical bases and a depot. The demand processes, costs, and transportation times are the same at both bases. Each base is assumed to follow the same (S,s) policy. Again, we could easily extend the results to the case where there are more than two bases; however, for ease of discussion we will describe in detail only the two base case. As an example of this case, the Air Force operates FB-111 aircraft mainly at two locations at which approximately the same removal rates are observed for most components.

An excellent survey of many papers on multi-echelon inventory systems is given by Clark [1]. Most of these papers deal with the periodic

review situation. Our work specifically extends the results given by Simon [6] and Kruse and Kaplan [4]. Their papers pertain to two-echelon systems in which the depot follows a continuous review (S,s) policy, but the bases are restricted to a continuous review (S,S-1) policy.

Before proceeding with the analysis, we pause to comment on the motivation for this research. This paper is part of a larger study which examines the impact on cost and performance of following different inventory policies in several real multi-item, multi-echelon systems [6]. The results developed here will be used to evaluate the inventory performance obtained using different values of the policy variables for each item at each location in one multi-item system. In particular, the probability distribution derived in this paper will be used to find the optimal values for the policy variable for each item in a real system when the objective is to select the inventory levels that minimize total expected base backorders subject to a constraint on system inventory investment. These values will then be compared to those obtained using several approximation methods. General conclusions will be drawn in that study concerning the quality and appropriateness of the several approximation methods for determining policy variable values for items having different demand and cost characteristics when different budget restrictions are imposed on the system.

II. SITUATION ONE--A LARGE NUMBER OF BASES

In this section we develop the probability distribution for the number of units backordered at a particular base at a random point in time when the number of bases in the system is large. The specific assumptions we make are as follows.

1. All bases and the depot follow a continuous review (S,s) policy, and all the bases use the same values for S and s .
2. No partial fill of base orders is permitted; that is, all $S-s$ units must be shipped simultaneously from the depot to satisfy a base order.
3. A stationary Poisson process generates demand at each base. Furthermore, the demand rate λ is the same at each base.
4. Demands occurring when a base has no on-hand inventory are back-ordered.
5. The depot-to-base transportation time T is constant and the same for all bases.
6. The depot resupply time R is constant.
7. The depot reorder point r_D is greater than or equal to minus one; the base reorder points are also greater than or equal to minus one.
8. The number of bases in the system is large.

We now discuss the implications of these assumptions.

Since the demands at the bases are generated one-at-a-time, the (S,s) policy is the same as a (Q,r) policy, where r indicates the reorder point and $Q = S - s$. Hence, we will refer to the base policy as a (Q,r) policy.

Next, we make an observation about the depot's inventory position. By assumption, a partial fill of a base's order is prohibited. Furthermore, we observed that our assumptions imply that all bases should follow an identical (Q, r) policy. It then follows that the depot inventory position should always be a multiple of Q . That is, there would be no advantage to having the inventory position be other than a multiple of Q since extra holding costs would be incurred without improving the chance of filling orders. Thus the depot always orders in multiples of Q ; that is, the depot reorder quantity, Q_D , can be expressed as

$$Q_D = M \cdot Q$$

for some $M = 1, 2, \dots$.

One can show that Assumptions 1 and 3 imply that the probability distribution for the depot inventory position is uniformly distributed over $J = \{r_D + Q, r_D + 2Q, \dots, r_D + Q_D\}$ [5].

Our last observation is a consequence of Assumption 8. Since the number of bases is large and the successive times between placing orders by a base for depot resupply form a renewal process, the order arrival process at the depot can be accurately approximated by a Poisson process [3].

Before proceeding with the derivation of the probability distribution of base backorders, let us introduce the nomenclature used in the remainder of the paper:

I_0 represents the depot inventory position at time $t-T-R$,

I^1 represents base j 's inventory position at time $t-T-R$,

I^2 represents base j 's inventory position at time $t-T$,

G represents the number of orders placed by all bases other than base j for depot resupply during $(t-T-R, t-T]$,

D represents the number of demands occurring during $(t-T-R, t-T]$ at base j ,

\tilde{D} represents the number of demands occurring during $(t-T, t]$ at base j ,

V represents the number of satisfied orders placed by base j on the depot during $(t-T-R, t-T]$ -- the orders are placed during $(t-T-R, t-T]$ and received at base j prior to t ,

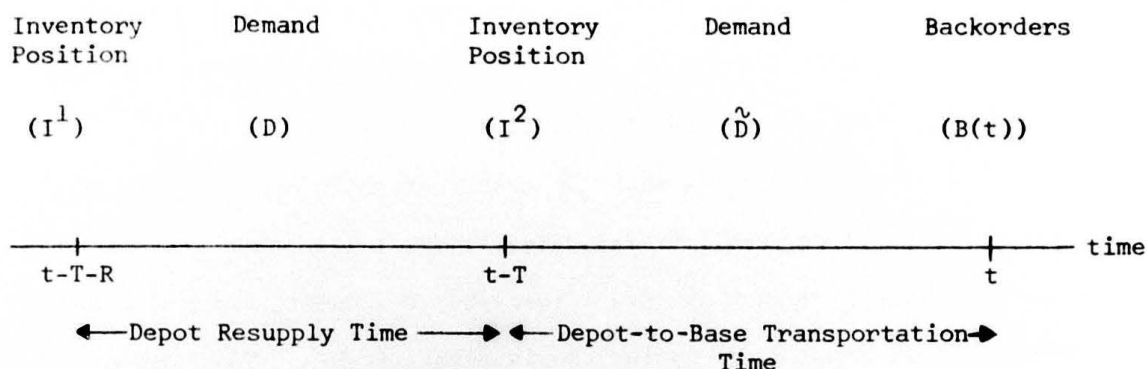
$$a(x, n) = e^{-x} x^n / n!,$$

$B(t)$ represents the number of backorders existing at base j at time t ,

U represents the number of orders placed on the depot during $(t-T-R, t-T]$ by base j that are unfilled at time t ,

and γ represents the arrival rate of orders at the depot from all bases except base j measured in orders per day.

The effect of time at base j is displayed in Figure 1.



Time Sequence of Events at Base j

Figure 1

We will consider the steady state behavior of this system. In particular, our interest is in the steady state distribution of the number of backorders at a base; that is, we want to find $P(B(t) = b)$ for $b = 0, 1, \dots$. To compute it, observe that any order placed by base j prior to time $t-T-R$ has been satisfied by time t . However, some orders placed on the depot during the interval $(t-T-R, t-T]$ by base j may not be satisfied by time t . This could cause backorders to exist at time t for some of the units demanded during $(t-T-R, t-T]$. Observe that the inventory available to satisfy demands during the interval $(t-T, t]$ is $I^2 - U \cdot Q$. If the demand at base j during $(t-T, t]$ exceeds this amount or this quantity is negative, then backorders will exist at time t at base j .

To compute $P(B(t) = b)$ we will first compute $P(U = u)$. We examine two cases. In the first case, which we call Case A, the total

demand on the depot during the interval $(t-T-R, t-T]$ does not exceed the available depot inventory (I_0) , so that $U = 0$. In the second case, which we call Case B, the total depot demand during $(t-T-R, t-T]$ exceeds I_0 , so that U may be positive.

First consider Case A, when the number of units ordered from the depot during the interval $(t-T-R, t-T]$ is not larger than the available depot inventory. Note that $[(D-I^1+Q+r)/Q]$ represents the number of orders placed by base j during the interval $(t-T-R, t-T]$, where $[k]$ represents the greatest integer less than or equal to k . Thus, in this case, $G + [(D-I^1+Q+r)/Q] \leq I_0/Q$. As we observed, in this situation all orders placed by base j on the depot prior to time $t-T$ will be satisfied by time t , and, therefore, $P(U = 0 | \text{Case A}) = 1$. Thus all shortages existing at time t at base j are due to demands placed at base j during $(t-T, t]$.

Then

$$\begin{aligned}
 P(B(t) = b; \text{Case A}) &= \sum_{i=r+1}^{r+Q} P(\tilde{D} = i+b | \text{Case A}; I^2 = i) \cdot P(I^2 = i; \text{Case A}) \\
 &= \sum_{i=r+1}^{r+Q} P(\tilde{D} = i+b) \cdot P(I^2 = i; \text{Case A}) \\
 &= \sum_{i=r+1}^{r+Q} a(\lambda T, i+b) \cdot P(I^2 = i; \text{Case A}), \text{ for } b \geq 1, \text{ (Eq. 1)}
 \end{aligned}$$

since \tilde{D} does not depend on events prior to time $t-T$. We will next find $P(I^2 = i; \text{Case A})$.

Suppose $I^2 = i$, $I^1 = k$, and $k \leq i$. Then demand at base j during $(t-T-R, t-T]$ must equal one of the numbers

0, $Q-(i-k)$, $Q-(i-k)+Q$, $Q-(i-k)+2Q, \dots$. That is, the demand at base j , n , must be an element of the set

$$N_1 = \{n: n = Q-(i-k) + \bar{n}Q, \bar{n} = 0, 1, 2, \dots, \text{ and } n \leq I_0/Q\} \cup \{0\}.$$

Next, suppose $I^2 = i < I^1 = k$. Then demand at base j during $(t-T-R, t-T]$ must equal one of the numbers $k-i$, $k-i+Q$, $k-i+2Q, \dots$, and, therefore, must be an element of the set

$$N_2 = \{n: n = k-i + \bar{n}Q, \bar{n} = 0, 1, 2, \dots, \text{ and } n \leq I_0/Q\}.$$

We will now find $P(I^2 = i; \text{Case A})$ by considering separately the cases where $I^2 = i \geq I^1 = k$ and $i < k$. Thus

$$\begin{aligned} P(I^2 = i; \text{Case A}) &= \sum_{i_0 \in J} \sum_{k=r+1}^i \sum_{n \in N_1} \sum_{m=0}^{i_0/Q - [(n-k+Q+r)/Q]} i_0/Q - [(n-k+Q+r)/Q] \\ &\quad P(I^2=i | I^1=k; G=m; D=n; I_0=i_0) \cdot P(I^1=k; G=m; D=n; I_0=i_0) \\ &+ \sum_{i_0 \in J} \sum_{k=i+1}^{r+Q} \sum_{n \in N_2} \sum_{m=0}^{i_0/Q - [(n-k+Q+r)/Q]} i_0/Q - [(n-k+Q+r)/Q] \\ &\quad P(I^2=i | I^1=k; G=m; D=n; I_0=i_0) \cdot P(I^1=k; G=m; D=n; I_0=i_0). \end{aligned}$$

(Eq. 2)

Observe that when $I^1 \leq I^2$

$$P(I^2=i | I^1=k; G=m; D=n; I_0=i_0) = \begin{cases} 0, & n \notin N_1 \quad (\text{and Case A conditions hold}) \\ 1, & n \in N_1 \quad (\text{and Case A conditions hold}); \end{cases}$$

also, when $I^1 > I^2$

$$P(I^2=i | I^1=k; G=m; D=n; I_0=i_0) = \begin{cases} 0, n \notin N_2 & (\text{and Case A conditions hold}) \\ 1, n \in N_2 & (\text{and Case A conditions hold}). \end{cases}$$

Furthermore, due to the independence of the random variables I^1 , G , D , and I_0 (see reference 5),

$$P(I^1=k; G=m; D=n; I_0=i_0) = P(I^1=k) \cdot P(G=m) \cdot P(D=n) \cdot P(I_0=i_0),$$

But $P(I^1=k) = 1/Q$ (see reference 2),

$P(G=m) = a(\gamma R, m)$, for $m = 0, 1, \dots$ (due to the basic assumptions),

$P(D=n) = a(\lambda R, n)$, for $n = 0, 1, \dots$ (since demand is Poisson distributed at base j),

and

$$P(I_0=i_0) = \frac{1}{M}, \text{ for } i_0 \in J \text{ (see reference 5).}$$

Hence $P(I^2 = i; \text{Case A})$ can be found by substituting the above probability expressions into Eq. 2. Furthermore, by substituting the resultant expression into Eq. 1 we have obtained $P(B(t) = b; \text{Case A})$.

Let us now consider Case B and determine the probability of observing b backorders at base j at time t given that the total number of units demanded from the depot during $(t-T-R, t-T]$ exceeds the available depot

inventory; that is, $G + [(D - I^1 + Q + r)/Q] > I_0/Q$. Observe that

$$B(t) = \max(0, \tilde{D} + U \cdot Q - I^2),$$

Then

$$\begin{aligned} P(B(t) = b; \text{Case B}) &= \sum_{i=r+1}^{r+Q} P(B(t) = b | \text{Case B}; I^2 = i) \cdot P(I^2 = i; \text{Case B}) \\ &= \sum_{i=r+1}^{r+Q} \sum_{u=0}^{[(i+b)/Q]} P(\tilde{D} = i + b - uQ) \cdot P(U = u | \text{Case B}; I^2 = i) \\ &\quad \cdot P(I^2 = i; \text{Case B}) \text{ for } b \geq 1. \end{aligned}$$

We now find $P(\tilde{D} = i + b - uQ)$, $P(U = u | \text{Case B}; I^2 = i)$ and $P(I^2 = i; \text{Case B})$.

Since \tilde{D} has a Poisson distribution,

$$P(\tilde{D} = i + b - uQ) = a(\lambda T, i + b - uQ).$$

Rather than directly computing $P(U = u | \text{Case B}; I^2 = i)$, we will find the distribution of the number of satisfied orders at time t that were placed during $(t - T - R, t - T]$ on the depot by base j ; that is, we will determine the conditional distribution for V .

We begin by evaluating

$$P(V = v | D = d; G = g; I^1 = k; I_0 = Qq; \text{Case B}).$$

The arrival process at base j is a Poisson process; furthermore, the

process generating depot orders from all other bases is approximately a Poisson process. From base j 's viewpoint, system arrivals consist of the demands placed at that base and depot orders placed by all other bases. The sequence in which orders are placed on the depot by base j and all other bases determines the number of satisfied base j depot orders at time t . If $D = d$ and $G = g$, then every ordering of the $d+g$ arrivals is equally likely to occur since the combined arrival process is a Poisson process (being the superposition of two Poisson processes--the base j arrival process and the depot order process from all other bases).

We will compute $P(V=v|D=d; G=g; I^1=k; I_0=qQ; \text{Case B})$ by considering two cases. In the first case, the last satisfied depot order is placed by a base other than j . Then if v orders are satisfied and $I^1 = k$, at least $vQ-(Q+r-k)$ and not more than $vQ-(Q+r-k)+Q-1$ demands must occur at base j prior to the arrival of the $(I_0/Q-v)$ -th order at the depot from a base other than j . In the second case, the last satisfied depot order is placed by base j .

Combining the above observations we see that

$$P(V=v|D=d; G=g; I^1=k; I_0=qQ; \text{Case B})$$

$$= \sum_{w=0}^{Q-1} \frac{\binom{d}{vQ-(Q-k+r)+w} \binom{g}{q-v-1}}{\binom{d+g}{vQ-(Q-k+r)+w+q-v-1}} \cdot \frac{g-(q-v-1)}{g+d-(vQ-(Q-k+r)+w+q-v-1)} \\ + \frac{\binom{d}{vQ-(Q-k+r)-1} \binom{g}{q-v}}{\binom{d+g}{vQ-(Q-k+r)+q-v-1}} \cdot \frac{d-(vQ-(Q-k+r)-1)}{g+d-(vQ-(Q-k+r)-1+q-v)} \quad (\text{Eq. 3})$$

whenever $v \leq q$; otherwise, the probability is 0. Also, we define

$$\binom{p}{z} = 0 \text{ if } z > p, z < 0, \text{ or } p < 0.$$

Now let us find

$$P(D=d; G=g; I^1=k; I_0=qQ; \text{Case B}).$$

This probability is equal to

$$\begin{aligned} & P(D=d; G=g; I^1=k; I_0=qQ | \text{Case B}; I^2 = i) \cdot P(\text{Case B}; I^2 = i) \\ &= P(\text{Case B}; I^2 = i | D=d; G=g; I^1=k; I_0=qQ) \cdot P(D=d; G=g; I^1=k; I_0=qQ) \\ &= \begin{cases} 0; & \text{if } [(d+Q+r-k)/Q] + g \leq I_0/Q = q \text{ or } d \notin N_1 \text{ when } i \geq k \\ & \text{and } d \notin N_2 \text{ when } i < k \\ 1 \cdot a(\lambda R, d) \cdot a(\gamma R, g) \cdot \frac{1}{Q} \cdot \frac{1}{M}; & \text{otherwise.} \end{cases} \end{aligned}$$

Consequently,

$$\begin{aligned} & P(D=d; G=g; I^1=k; I_0=qQ | \text{Case B}; I^2 = i) \\ &= \begin{cases} 0; & [(d+Q+r-k)/Q] + g \leq q \text{ or } d \notin N_1 \text{ when } i \geq k \\ & \text{and } d \notin N_2 \text{ when } i < k \\ a(\lambda R, d) \cdot a(\gamma R, g) \cdot \frac{1}{Q} \cdot \frac{1}{M} \cdot \frac{1}{P(\text{Case B}; I^2 = i)}; & \text{otherwise} \end{cases} \end{aligned}$$

Now observe that $V = v$ if and only if $U = [(d+Q+r-k)/Q] - v$ when $D = d, G = g, I^1 = k$, and $I_0 = qQ$ in Case B. Therefore

$$\begin{aligned}
P(U=u|D=d; G=g; I^1=k; I_0=qQ; \text{Case B}) \\
= P(V = [(d+Q+r-k)/Q]-u|D=d; G=g; I^1=k; I_0=qQ; \text{Case B}).
\end{aligned}$$

Then using Eq. 3 we may find the conditional probability for the number of unsatisfied depot orders at time t for orders placed by base j during $(t-T-R, t-T]$.

Upon combining the previous results we see that

$$\begin{aligned}
P(U=u; D=d; G=g; I^1=k; I_0=qQ | \text{Case B}) \\
= P(U=u|D=d; G=g; I^1=k; I_0=qQ; \text{Case B}) \cdot P(D=d; G=g; I^1=k; I_0=qQ | \text{Case B}),
\end{aligned}$$

and

$$\begin{aligned}
P(U=u | \text{Case B}; I^2 = i) &= \sum_{q=r_D/Q+1}^{r_D/Q+M} \sum_{k=r+1}^{r+Q} \sum_{d=0}^{\infty} \sum_{g=q-[(d+Q+r-k)/Q]+1}^{\infty} \\
&P(V = [(d+Q+r-k)/Q]-u | D=d; G=g; I^1=k; I_0=qQ; \text{Case B}) \\
&\cdot P(D=d; G=g; I^1=k; I_0=qQ | \text{Case B}; I^2 = i).
\end{aligned}$$

The final probabilities that we must compute to complete the calculation of $P(B(t) = b; \text{Case B})$ are $P(I^2 = i; \text{Case B})$.

$P(I^2 = i; \text{Case B})$ can be derived in virtually the same way we previously derived $P(I^2 = i; \text{Case A})$. It is easy to show that

$$\begin{aligned}
P(I^2 = i; \text{Case B}) &= \sum_{i_0 \in J} \sum_{k=r+1}^i \sum_{n \in N_1} \sum_{m=(i_0/Q)-[(n-k+Q+r)/Q]+1}^{\infty} \\
&\quad \frac{1}{Q} \cdot a(\gamma R, m) \cdot a(\lambda R, n) \cdot \frac{1}{M} \\
&+ \sum_{i_0 \in J} \sum_{k=r+1}^{r+Q} \sum_{n \in N_2} \sum_{m=(i_0/Q)-[(n-k+Q+r)/Q]+1}^{\infty} \\
&\quad \frac{1}{Q} \cdot \frac{1}{M} \cdot a(\gamma R, m) \cdot a(\lambda R, n).
\end{aligned}$$

Thus we have found $P(B(t) = b; \text{Case A})$ and $P(B(t) = b; \text{Case B})$.

Since the two cases form a partition,

$$P(B(t) = b) = P(B(t) = b; \text{Case A}) + P(B(t) = b; \text{Case B})$$

for $b \geq 1$ and

$$P(B(t) = 0) = 1 - \sum_{b \geq 1} P(B(t) = b).$$

III. SITUATION TWO--TWO BASES AND A DEPOT

In this section we examine a system consisting of a depot and two identical bases. The system's operation is described in Section I, Assumptions 1)-7) stated in the previous section are still appropriate; however, we assume in this case that there are only two bases in the system. Our objective, as before, is to determine the probability distribution for the number of units backordered at a base at an arbitrary point in time.

The desired probability distribution will be developed by again examining two separate cases. In Case A, depot demand during $(t-T-R, t-T]$ is assumed not to exceed I_0 by time $t-T$. In Case B, depot demand exceeds I_0 during $(t-T-R, t-T]$. Depot orders placed by a base may not be satisfied by time t in the latter case. To find the probability distribution for the number of backordered units at a base at time t , we examine the arrival sequences at the bases. Information relating to the inventory position at each base and the depot at time $t-T-R$ will be combined with knowledge of the arrival sequence at the bases. In the previous section, G --the number of orders placed on the depot by other bases--was Poisson. Here it is not and our development parallels that of Section II except for this difference.

Without loss of generality, we will determine the desired probability distribution for base 1. Subscripts in this section refer to a particular base. In Case A we assume that

$$[(D_1 - I_1^1 + Q + r)/Q] + [(D_2 - I_2^1 + Q + r)/Q] \leq I_0/Q;$$

that is, the total depot demand during $(t-T-R, t-T]$ does not exceed the inventory available to meet that demand, I_0 .

Since all demands placed on the depot during the period $(t-T-R, t-T]$ are satisfied by time t in Case A, all backorders at base 1 at time t are a result of demands occurring during the interval $(t-T, t]$. Thus

$$\begin{aligned} P(B_1(t) = b; \text{Case A}) &= \sum_{i=r+1}^{r+Q} P(\tilde{D}_1 = i+b | \text{Case A}; I_1^2 = i) \cdot P(I_1^2 = i; \text{Case A}) \\ &= \sum_{i=r+1}^{r+Q} a(\lambda T, i+b) \cdot P(I_1^2 = i; \text{Case A}), \quad \text{for } b \geq 1. \end{aligned}$$

Using the same logic as given earlier we see that

$$\begin{aligned} P(I_1^2 = i; \text{Case A}) &= \sum_{i_0 \in J} \sum_{k=r+1}^i \sum_{n \in N_1} \sum_{j=r+1}^{Q+r} \sum_{d \in F} \\ &\quad P(I_1^2 = i | I_1^1 = k; D_2 = d; D_1 = n; I_2^1 = j; I_0 = i_0) \\ &\quad \cdot P(I_1^1 = k; D_2 = d; D_1 = n; I_2^1 = j; I_0 = i_0) \\ &+ \sum_{i_0 \in J} \sum_{k=i+1}^{r+Q} \sum_{n \in N_2} \sum_{j=r+1}^{r+Q} \sum_{d \in F} \quad (\text{Eq. 4}) \\ &\quad P(I_1^2 = i | I_1^1 = k; D_2 = d; D_1 = n; I_2^1 = j; I_0 = i_0) \\ &\quad \cdot P(I_1^1 = k; D_2 = d; D_1 = n; I_2^1 = j; I_0 = i_0), \end{aligned}$$

where N_1 and N_2 have the same meanings as stated in Section II, and

$$F = \{d: d = 0, 1, \dots; [(n-k+Q+r)/Q] \cdot Q + [(d-j+Q+r)/Q] \cdot Q \leq i_0\}.$$

But when $I_1^1 \leq I_1^2$,

$$P(I_1^2=i | I_1^1=k; D_2=d; D_1=n; I_2^1=j; I_0=i_0) = \begin{cases} 1; n \in N_1 & (\text{Case A holds}) \\ 0; & \text{otherwise;} \end{cases}$$

and when $I_1^1 > I_2^1$,

$$P(I_1^2=i | I_1^1=k; D_2=d; D_1=n; I_2^1=j; I_0=i_0) = \begin{cases} 1; n \in N_2 & (\text{Case A holds}) \\ 0; & \text{otherwise.} \end{cases}$$

By independence,

$$\begin{aligned} P(I_1^1=k; D_2=d; D_1=n; I_2^1=j; I_0=i_0) &= P(I_1^1=k) \cdot P(D_2=d) \cdot P(D_1=n) \cdot P(I_2^1=j) \cdot P(I_0=i_0) \\ &= \frac{1}{Q^2} \cdot \frac{1}{M} \cdot a(\lambda R, n) \cdot a(\lambda R, d). \end{aligned}$$

Upon substituting these results into Eq. 4 we have $P(I_1^2 = i; \text{Case A})$.

In Case B, total depot demand during $(t-T-R, t-T]$ exceeds the available depot inventory; that is,

$$[(D_1 - I_1 + Q + r)/Q] + [(D_2 - I_2 + Q + r)/Q] > I_0/Q.$$

Then

$$\begin{aligned} P(B_1(t) = b; \text{Case B}) &= \sum_{i=r+1}^{r+Q} \sum_{u=0}^{[(i+b)/Q]} P(\tilde{D}_1 = i+b-uQ) \cdot P(U_1 = u | \text{Case B}; I_1^2 = i) \\ &\quad \cdot P(I_1^2 = i; \text{Case B}), \quad b \geq 1. \end{aligned}$$

We will now find $P(\tilde{D}_1 = i+b-uQ)$, $P(U_1 = u | \text{Case B}; I_1^2 = i)$, and $P(I_1^2 = i; \text{Case B})$.

Since demand is Poisson distributed, $P(\tilde{D}_1 = i+b-uQ) = a(\lambda T, i+b-uQ)$.

As before, instead of determining $P(U_1 = u | \text{Case B}; I_1^2 = i)$ directly, we will first find $P(V_1 = v | D_1 = d_1; D_2 = d_2; I_1^1 = k; I_2^1 = j; I_0 = i_0; \text{Case B})$. As we saw earlier, the last satisfied depot order could have been placed by either base. Then conditioning on which base placed the last satisfied depot order, we see that

$$\begin{aligned}
 &P(V_1 = v | D_1 = d_1; D_2 = d_2; I_1^1 = k; I_2^1 = j; I_0 = i_0; \text{Case B}) \\
 &= \sum_{w=0}^{Q-1} \left\{ \frac{\binom{d_1}{vQ-(Q+r-k)-1} \binom{d_2}{\bar{v}Q-(Q+r-j)+w}}{\binom{d_1+d_2}{(v+\bar{v})Q-2(Q+r)+k+j+w-1}} \cdot \frac{d_1 - (vQ - (Q+r-k) - 1)}{d_1 + d_2 - ((v+\bar{v})Q - 2(Q+r) + k + j + w - 1)} \right. \\
 &\quad \left. + \frac{\binom{d_1}{vQ-(Q+r-k)+w} \binom{d_2}{\bar{v}Q-(Q+r-j)-1}}{\binom{d_1+d_2}{(v+\bar{v})Q-2(Q+r)+k+j+w-1}} \cdot \frac{d_2 - (\bar{v}Q - (Q+r-j) - 1)}{d_1 + d_2 - ((v+\bar{v})Q - 2(Q+r) + k + j + w - 1)} \right\}, \quad (\text{Eq. 5})
 \end{aligned}$$

where $\bar{v} = (i_0/Q) - v$. This result is based on the observation that each sequence of total customer demands occurring at the bases--the superposition of the two Poisson processes--is equally likely to occur given a fixed number of arrivals at each base.

Knowing $P(V_1 = v | D_1 = d_1; D_2 = d_2; I_1^1 = k; I_2^1 = j; I_0 = i_0; \text{Case B})$ is equivalent to knowing the probability distribution for the number of unsatisfied depot orders placed by base 1 during $(t-T-R, t-T]$.

Now

$$P(U_1 = u | \text{Case B}; I_1^2 = i)$$

$$= \sum_{i_0 \in J} \sum_{k=r+1}^{r+Q} \sum_{j=r+1}^{r+Q} \sum_{d_1=uQ-(Q+r-i)}^{i_0+k-r-1+uQ} \sum_{d_2=i_0-[(d_1-k+Q+r)/Q] \cdot Q-(Q+r-j)+1}^{\infty}$$

$$P(V_1 = [(d_1-k+Q+r)/Q]-u | D_1=d_1; D_2=d_2; I_1^1=k; I_2^1=j; I_0=i_0; \text{Case B})$$

$$\cdot P(D_1=d_1; D_2=d_2; I_1^1=k; I_2^1=j; I_0=i_0 | \text{Case B}; I_1^2 = i),$$

where

$$P(D_1=d_1; D_2=d_2; I_1^1=k; I_2^1=j; I_0=i_0 | \text{Case B}; I_1^2 = i)$$

$$= \begin{cases} a(\lambda R, d_1) \cdot a(\lambda R, d_2) \cdot \frac{1}{Q^2} \cdot \frac{1}{M} \cdot \frac{1}{P(\text{Case B}; I_1^2 = i)}; & \text{when Case B holds and} \\ & d_1 \in N_1 \text{ when } i \geq k \text{ and } d_1 \in N_2 \text{ when } i < k \\ 0; & \text{otherwise} \end{cases}$$

(N_1 and N_2 have the same interpretation as given in Section II), and

where Eq. 5 is used to evaluate

$$P(V_1 = [(d_1-k+Q+r)/Q]-u | D_1=d_1; D_2=d_2; I_1^1=k; I_2^1=j; I_0=i_0; \text{Case B})$$

whenever $d_2 \geq 0$; if $d_2 < 0$, this probability is 0. We will show how to compute $P(I_1^2 = i; \text{Case B})$.

Following the same reasoning as used previously for Case A, we can show that

$$\begin{aligned}
P(I_1^2 = i; \text{Case B}) = & \sum_{i_0 \in J} \sum_{k=r+1}^i \sum_{n \in N_1} \sum_{j=r+1}^{r+Q} \sum_{d \in H} \frac{1}{Q^2} \cdot \frac{1}{M} \cdot a(2\lambda R, n+d) \\
& + \sum_{i_0 \in J} \sum_{k=i+1}^{r+Q} \sum_{n \in N_2} \sum_{j=r+1}^{r+Q} \sum_{d \in H} \frac{1}{Q^2} \cdot \frac{1}{M} \cdot a(2\lambda R, n+d),
\end{aligned}$$

where

$$H = \{d: d = 0, 1, \dots, [(n-k+Q+r)/Q] \cdot Q + Q(d-j+Q+r)/Q \cdot Q > i_0\}.$$

Upon combining the above results we have shown how to determine $P(B_1(t) = b; \text{Case B})$. As before, since the two cases form a partition,

$$P(B_1(t) = b) = P(B_1(t) = b; \text{Case A}) + P(B_1(t) = b; \text{Case B})$$

when $b \geq 1$ and

$$P(B_1(t) = 0) = 1 - \sum_{b \geq 1} P(B_1(t) = b).$$

IV. CONCLUDING COMMENTS

In this paper, we have examined a two-echelon inventory system in which each location follows a continuous review (S,s) policy. The objective of the paper was to show how to express the probability distribution for the number of units backordered at any base at any arbitrary point in time. Two cases were studied. In the first case, we assumed there were a large number of identical bases and a depot in the system; in the second case, the system consisted of a depot and two identical bases.

The results stated in Section III can be extended to the case where there are any finite number of bases in the system. We studied the case where there were two bases only for ease of exposition. There is no theoretical difficulty hindering the development of the probability distributions when the number of bases considered exceeds two. Furthermore, a similar analysis can also be carried out for the situation in which the λ_i , S_i , and s_i differ between the two bases. To accomplish this, we drop assumption 2 and employ the methods used in chapter 5 of reference 8 and the basic approach developed in Section III. The probability distribution $P(B(t) = b)$ is somewhat more tedious to calculate in the general case; however, there are no theoretical problems that need to be overcome.

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the number of units required to raise the depot's inventory position to a prespecified level. Thus all locations follow a continuous review (S,s) policy. All excess demand is assumed to be backordered.

Our main objective is to derive the probability distribution for the number of backordered units at a base at an arbitrary point in time given an item follows a known (S,s) policy at each base and the depot. The demand process at each base is assumed to be a stationary Poisson process. The analysis is carried out for two cases. In the first case, we assume the system consists of a large number of bases; in the second case, we assume there are two bases in the system. To simplify the discussion, we assume in both cases that all bases follow the same (S,s) policy.

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